Working Paper ?? / 2023

The economics of sea-level rise: theoretical considerations

Part X: Stylized models of coasts

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August 14, 2023

Abstract The working paper is a accompanying script for a courde on Coastal impact and adaptation modelling I gave in 2023 at GCF. Its a working document which will be updated regularly during the course and transformed into a polished script after the course. This part of a series of working papers deals with stylized models of coastal landscapes, e.g. stylized models of flood plains etc.

Keywords Coastal impacts and adaptation · Sea-level rise · Profiles

1 Basics

Some basic definitions:

Definition 1.1 – $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{B}$ stand for the natural, integer and real numbers and for boolean values.

- $\mathbb{Z}_+, \mathbb{R}_+$ stand for the postive integer and real numbers (does not include zero).
- $\mathbb{Z}_{+,0}, \mathbb{R}_{+,0}$ stand for the postive integer and real numbers including zero.
- An infinite discrete (regular) grid is the set $G^\infty = \mathbb{Z} \times \mathbb{Z}$.
- A finite discrete (regular) grid $G^{n\times m}$ is a finite subset of G^∞, i.e. $G^{(n,m)} = [0, \dots, n-1] \times [0, \dots, m-1]$. Here n and m stand for the number of columns and rows and it is said the G has the dimension $n \times m$.
- Note that $\mathbb{N} \times \mathbb{N} \subset \mathbb{Z} \times \mathbb{Z}$ is still an infinite grid.
- The elements $(x, y) \in G$ (G without any superscript stands for both infinite and finite discrete grids) are called grid cells (synonym is grid points).
- We adopt here the technical notion inspired by the image processing: the upper left corner of any finite grid with dimension $n \times m$ is is the grid cell (0,0), the lower left corner is the grid cell $(0, m-1)$ and the upper right corner is the grid cell $(n-1, 0)$ (Fig. 1).
- We further denote for a finite discrete grid $G^{n\times m} = [0, \dots, n-1] \times [0, \dots, m-1]$ the indices function as follows:

$$cind_G(x,y) = \begin{cases} \varnothing & \text{if } x < 0 \text{ or } x \geq n-1 \\ \varnothing & \text{if } y < 0 \text{ or } y \geq m-1 \\ (x,y) & \text{otherwise} \end{cases}$$

  For an infinite discrete grid G^∞ the indices function is defined similarly (the $\varnothing$ cases do not exist then).
- For a discrete grid G the 4-neighbourhoud (Fig. 2) of a grid cell (x, y) is defined as

$$M^4_G(x,y) = \{cind_G(x-1,y)\} \cup \{cind_G(x+1,y)\} \cup \{cind_G(x,y-1)\} \cup \{cind_G(x,y+1)\}$$

  The pair (x, y) itself is not part of its own neighbourhoud. It should be noted that $\forall G, x, y : 0 \leq \left|M^4_G\right| \leq 4$ The 4-neighbourhoud is also referred to as Von Neumann neighborhood Wilson and Ritter (2000).
- In the same way the 8-neighbourhoud (Fig. 2) of a grid cell $(x, y) \in G$ is defined as

$$\begin{aligned} M^8_G(x,y) = &\{cind_G(x-1,y)\} \cup \{cind_G(x+1,y)\} \cup \{cind_G(x,y-1)\} \cup \{cind_G(x,y+1)\} \cup \{cind_G(x-1,y-1)\} \\ &\cup \{cind_G(x+1,y-1)\} \cup \{cind_G(x-1,y+1)\} \cup \{cind_G(x+1,y+1)\} \end{aligned}$$

  Agagin, the pair (x, y) itself is not part of its own neighbourhoud. It should be noted that $\forall G, x, y : 0 \leq \left|M^8_G\right| \leq 8$

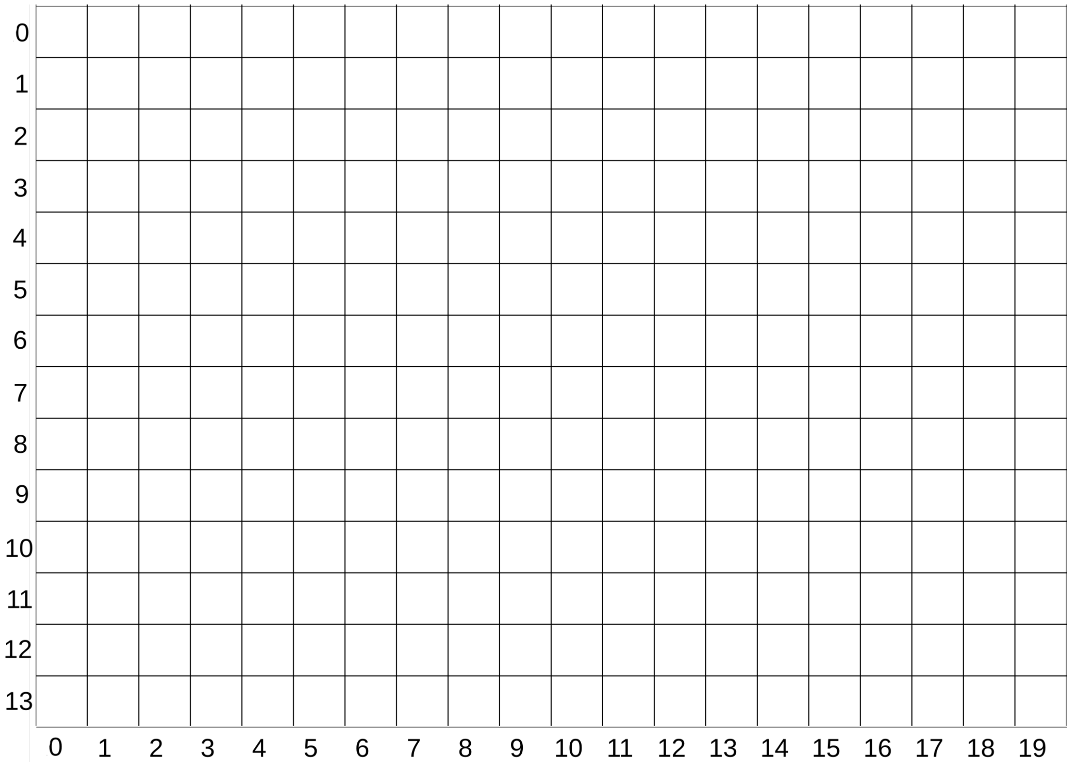

Fig. 1 A finite grid with dimension 20×14.

– For any grid G the neighbourhoud with radius r (see Fig. 2) of a grid cell (x, y) is defined as

$$N_G^r(x,y) = \left(\bigcup_{k=-r}^{r} \bigcup_{l=-r}^{r} cind_X(x+k, y+l) \right) \setminus (x,y)$$

From this definition and the definition above it follows that $N_G^1(x,y) = M_G^8(x,y)$.

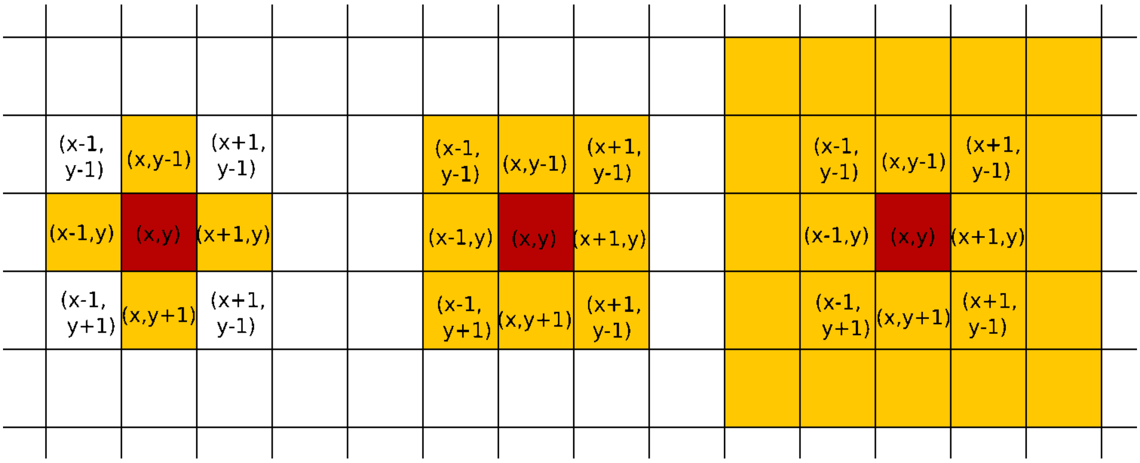

Fig. 2 Illustration of the 4-neighbourhoud (left) and the 8-neighbourhoud (middle) and the Neighbourhoud with radius 2 of a grid cell.

– A gridded dataset on a (finite or infinite) discrete grid G is a mapping

$$d : G \to M \cup \{\perp\}$$

where M is the set of possile data values and $\perp$ denotes a special value (interpreted as no data).

- In the same way a gridded multivariate dataset on a (finite or infinite) discrete grid G is a mapping

$$d : G \to (M_1 \cup \{\perp\}) \times \ldots \times (M_k \cup \{\perp\})$$

where $k \geq 1$ is the dimension of the the dataset.
- for each gridded dataset the function *grid* strips the grid from the dataset
- For a gridded dataset *data* on a grid $G^{n\times m}$ the 4-neighbourhood values are defined as

$$data^4_G(x, y) = \{data(i, j) \mid (i, j) \in M^4_G(x, y)\}$$

and in a similar way the 8-neighbourhood values ($data^8_G(x, y)$) and the neighbourhood values with radius r ($N^r_{data,G}(x, y)$) are defined.
- For a gridded dataset *data* on a grid $G^{n\times m}$ two grid cells (x_s, y_s) and (x_e, y_e) are defined to be 4-connected if there exists a sequence of grid cells $p = (gc_1, \ldots, gc_n) = ((x_1, y_1), \ldots, (x_n, y_n))$ such that
  - $gc_1 = (x_s, y_s)$
  - $gc_n = (x_e, y_e)$
  - $i \neq j \;\to gc_i \neq gc_j$
  - $\forall\; gc_i = (x_i, y_i) : data(x_i, y_i) \neq \perp$
  - $\forall\; gc_i = (x_i, y_i)$ with $i > 1 : (x_i, y_i) \in M^4_G(x_{i-1}, y_{i-1})$

  The sequence p is called a 4-connected path from $g_1 = (x_s, y_s)$ to $g_n = (x_e, y_e)$. In a similar way two grid cells (x_s, y_s) and (x_e, y_e) are defined to be 8-connected.
- For a gridded dataset *data* on a grid $G^{n\times m}$ and a predicate $pred : (M \cup \{\perp\})^k \to Boolean$ two grid cells (x_s, y_s) and (x_e, y_e) are defined to be *pred*-constrained 4-connected if there exists a sequence of grid cells $p = (gc_1, \ldots, gc_n)$ such that
  - p is a four-connected path from gc_1 to gc_n.
  - $\forall\; gc_i = (x_i, y_i) : pred(x_i, y_i) = true$

  The sequence p is called a *pred*-constrained 4-connected path from (x_s, y_s) to (x_e, y_e). In a similar way two grid cells (x_s, y_s) and (x_e, y_e) are defined to be *pred*-constrained 8-connected.
- For any path $p = (gc_1, \ldots, gc_n)$ the value $max_{data}(p) = max\{data(gc_1), \ldots, data(gc_n)\}$ is called the maximal data value of p.

Example 1.1 A discrete digital elevation model (DEM) on a (finite) discrete grid G is a gridded dataset

$$DEM : G \to \mathbb{R}^{\perp}$$

where $R_{\perp}$ is a shortcut for $\mathbb{R} \cup \{\perp\}$. The values $DEM(x, y)$ are interpreted as the elevation of the grid cell at position (x, y) relative to some elevation reference. If $DEM(x, y) = \perp$, the grid cell does not represent land (but rather ocean). The nature of the DEM can be further specified, e.g. as a digital surface elevation model or a digital ground elevation model.

Example 1.2 A discrete digital coastal surface elevation and distance model $DSEDM$ on a (finite) discrete grid G is a gridded dataset

$$DSEDM : G \to \mathbb{R}^{\perp} \times \mathbb{R}^{\perp}_{+}$$

where The values $DSEDM(x, y)$ are pairs (e, d) where e is interpreted as the elevation of grid cell (x, y) and d is interpreted as the distance of grid cell (x, y) to the coastline.

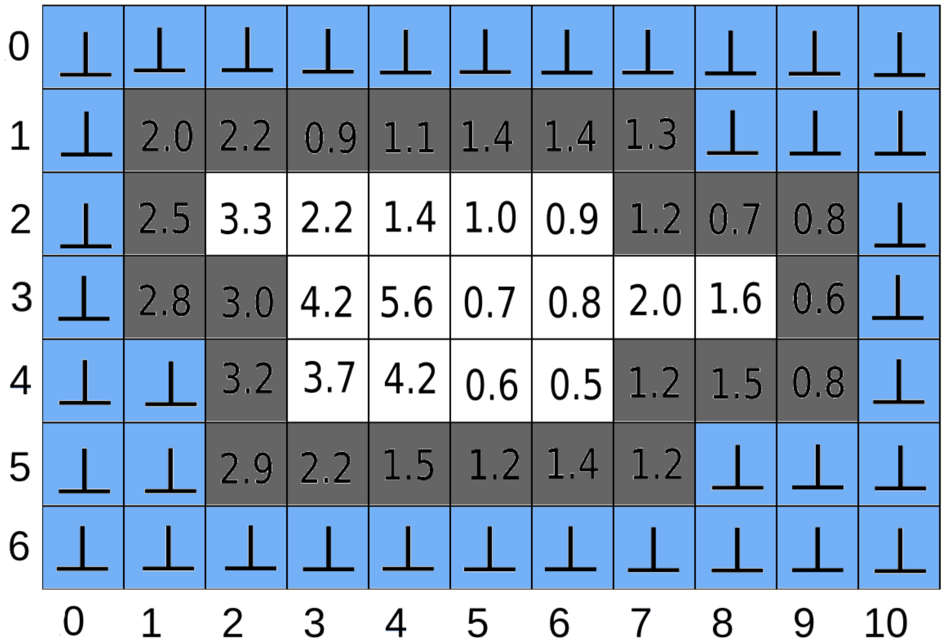

Fig. 3 Illustration of an DEM with water ($\perp$ with blue shaded background) and its interpretation as CM. Coastline grid cells have grey shaded background. For these grid cells the second dimension in a coastal model would be true, for all other grid cells false.

2 Floodplain Profiles

Some basic definitions (coastal model):

Definition 2.1 In a discrete DEM a grid cell $g = (x, y)$ is called coastline iff $DEM(x, y) \neq \perp \ \wedge \ \perp \in DEM^8_G(x, y)$. That is, the coastline is the set of grid cells that contain a data value and at least one no data value in their 8-neighbourhood.

Fig. 3 illustrates the concept. An algorithm to detect coastline is easy to sketch:

```
extractCoastline (dem: DEM): Dataset
  var dat: Dataset
  for ((x,y):grid(dem))
    if (dem(x,y) != nodata(dem))
      nh8 = Neighbourhood8(dem,x,y)
      for ((x2,y2):nh8)
        if(dem(x2,y2) == nodata(dem))
          insert(dat,x,y,dem(x,y))
          break
        end
      end
    end
  end
  return dat
end
```

In reality such an operation is much more difficult to implement: one might want to include additional data (e.g. ocean-connected water data for modelling river mouths and deltas) and the DEM might be huge and be provided in pieces (tiles etc). An extracted coastline can be combined with a DEM into a coastal model:

Definition 2.2 A CM cm is a two-variate dataset (as defined before)

$$cm : G \rightarrow \mathbb{B} \times \mathbb{R}^{\perp}$$

where G is a finite grid. In a coastal model the value $cm(x,y) = (c,e)$ is interpreted as follows: b is true if and only if the grid cell is coastline and e decodes the elevation. For a CM cm the following two projections are defined:

$$\begin{aligned} coastline : G \to \mathbb{B} \\ coastline(x,y) &= c \iff cm(x,y) = (c,e) \\ elevation : G \to \mathbb{R}^{\perp} \\ elevation(x,y) &= e \iff cm(x,y) = (c,e) \end{aligned}$$

This definition can be generalized into an extended CM with arbitrary many datasets involved:

Definition 2.3 An (extended) CM cm is a multi-variate dataset (as defined before)

$$cm : G \to \mathbb{B} \times \mathbb{R} \times (\mathbb{R}^{\perp})^n$$

where G is a finite grid and $n \in \mathbb{N}$ with $n \geq 0$ and further $cm(x,y) = (c, \perp, \vec{d}) \Rightarrow \vec{d} = \vec{\perp}$. In a coastal model the values c and e in $cm(x,y) = (c,e,\vec{d})$ are interpreted as before. The projection functions remain and I assume that there is a projection function (with a meaningful name) for each dimension of $\vec{d}$.

An (extended) CM can be seen as a combination of several gridded datasets (all operating on the same grid). The elevation dataset is given explicitly with the requirement that if elevation is no data then all other dimensions should also be no data.

Example 2.1 A simple coastal model for flood assessment could be defined as an extended CM $cm : G \to \mathbb{B} \times \mathbb{R}^{\perp} \times (\mathbb{R}^{\perp})^4$ with the following projections:

$$\begin{aligned} coastline : G \to \mathbb{B} \\ coastline(x,y) &= c \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \\ elevation : G \to \mathbb{R}^{\perp} \\ elevation(x,y) &= e \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \\ area : G \to \mathbb{R}^{\perp}_{+,0} \\ area(x,y) &= d_1 \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \\ population : G \to \mathbb{R}^{\perp}_{+,0} \\ population(x,y) &= d_2 \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \\ assets : G \to \mathbb{R}^{\perp}_{+,0} \\ assets(x,y) &= d_3 \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \\ distance : G \to \mathbb{R}^{\perp} \\ distance(x,y) &= d_4 \iff cm(x,y) = (c,e,d_1,d_2,d_3,d_4) \end{aligned}$$

In the remainder onf the paper I do not distinguish between CM and extended CM anymore. The involved dimensions and projections should be clear from the context.

Starting from a CM cm coastal zones and flood plains can be defined.

Definition 2.4 For a CM cm the coastal zone with elevation threshold θ_{el} ($cz(cm, \theta_{el})$) is defined as

$$cz(cm,\theta_{el}) : G \to \mathbb{R}^{\perp}$$

$$cz(cm,\theta_{el})(x,y) = \begin{cases} el & \text{if } elevation(x,y) = el \text{ and } \exists(x_2,y_2) \text{ s.t. } coastline(x_2,y_2) = true \text{ is} \\ & \leq_{\theta_{el}} \text{-constrained 8-connected to } (x,y) \\ & \text{where } \leq_{\theta_{el}}(e) = e \leq \theta_{el} \\ \perp & \text{otherwise} \end{cases}$$

That is, $cz(cm, \theta_{el})$ is a dataset consisting from the same grid as the underlying CM cm mapping the original elevation value to all grid cells that are 8-connected to the coastline with a path for that all grid cells on the path have elevation of at most θ_{el}.

There are two special coastal zones with elevation threshold: the low elevation coastal zone (LECZ) ist the coastal zone with elevation threshold 10.0m and the extended low elevation coastal zone (ELECZ) ist the coastal zone with elevation threshold 20.0m. The coastal zones itself can be made part of the CM be introducing dimensions and projections for them.

2.1 Hydrologic connectivity

In a CM the elevation of grid cells does not determine their exposure to flooding. It is rather the hydrological connectivity that defines if a grid cell can be flooded by an extreme water level event. For instance, in Fig. 3 grid cell $(5, 4)$ contains an elevation value of 0.6. However, all paths from $(5, 4)$ to coastline grid cells contain grid cells with higher elevation.

Definition 2.5 In a CM cm the hydrologic connectivity is for a grid cell (x, y) is defined as

$$hc_{cm} : G \rightarrow \mathbb{R}^{\perp}$$
$$hc_{cm}(x,y) = \begin{cases} \perp & \text{if } CM(x,y) = (cl, \perp) \\ \min\{\max_{elevation}(p) : p \text{ path from } (x,y) \text{ to a coastline grid cell}\} & \text{otherwise} \end{cases}$$

That is, the hydrologic connectivity is the minimal maximal elevation on any path from the grid cell to the coastline. In order to be able to reach grid cell (x, y) an extreme water level event with must have at least this water level.

Lemma 2.1 *For any CM cm the hydrologic connectivity is a function, that is* $hc_{cm}(x, y) = e_1$ *and* $hc_{cm}(x, y) = e_2$ *implies* $e_1 = e_2$.

The lemma above states that the hydrological connectivity of a grid cell is unique. Its proof is left as an exercise.

2.2 Hypsometric profile

The hypsometric profile of a CM is a stylized model of the coastal plain that allows for simple computations of exposure, flood damages and adaptation.

Definition 2.6 Given a CM
$$cm : G \rightarrow \mathbb{B} \times \mathbb{R}^{\perp} \times (\mathbb{R}^{\perp})^n$$
a (discrete) hypsometric profile of cm is a function

$$dhsp_{cm} : \mathbb{R} \rightarrow \mathbb{R}^n_+$$
$$dhsp_{cm}(e) = \sum_{\substack{(x,y) \\ hc_{cm}(x,y) \leq e \\ cm(x,y)=(b,z,\vec{d})}} \vec{d}$$

In the definition above I use "discrete" as in general the function hsp_{cm} is not continuous. It can be made continuous by interpolation between existing values for e. Linear interpolation is widely used.

Definition 2.7 Given a CM
$$cm : G \rightarrow \mathbb{B} \times \mathbb{R}^{\perp} \times (\mathbb{R}^{\perp})^n$$
a (partial linear) hypsometric profile of cm is a function

$$hsp_{cm} : \mathbb{R} \rightarrow \mathbb{R}^n_+$$
$$hsp_{cm}(e) = \begin{cases} \vec{0} & \text{if } e < \min\{z : \exists(x,y) : hc_{cm}(x,y) = z\} \\ dhsp_{cm}(m) & \text{if } e > m \\ dhsp_{cm}(e) & \text{if } \exists(x,y) : hc_{cm}(x,y) = e \\ \frac{dhsp_{cm}(e_2) - dhsp_{cm}(e_1)}{e_2 - e_1} * (e - e_1) + dhsp_{cm}(e_1) & \text{otherwise} \end{cases}$$

$$\text{where } m = \max\{z : \exists(x, y) : hc_{cm}(x, y) = z\}$$
$$e_2 = \min\{z : z > e \text{ and } \exists(x, y) : hc_{cm}(x, y) = z\}$$
$$e_1 = \max\{z : z < e \text{ and } \exists(x, y) : hc_{cm}(x, y) = z\}$$

The function is continous in the interval $[m, \infty)$. In implemented models that build upon hypsometric profile there migth be an artifical grid cell added that maps all exposure data sets to zero. For instance, the DEM in Fig. 3 maps the grid cells to elvation values rounded to one digit with minimum elevation value 0.5. An artifical grid cell might be added with elevation 0.4 that maps all other datasets associated with this DEM to zero, in particular such an artifical grid cell has no area. By this addition the (partial linear) hypsometric profile becomes a continous function on $(-\infty, \infty)$.

Despite the continousity there are a few pleasant properties of hsp_{cm}. In particular it is non-decreasing and unique for a given CM cm.

In the remainder of this paper I will use hypsometric profile synonym for (partial linear) hypsometric profile.

Acknowledgements No Acknowledgements.

Acronyms

CM coastal model. 4–7

DEM digital elevation model. 3, 4, 7

ELECZ extended low elevation coastal zone. 6

LECZ low elevation coastal zone. 6

References

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