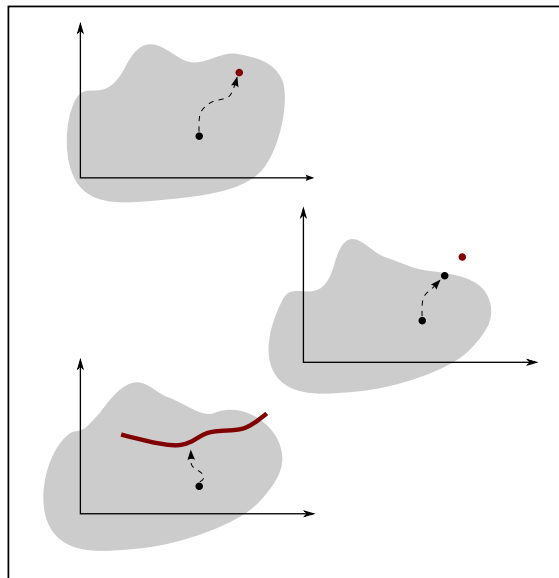


**Price Dynamics Via Expectations,
and the Role of Money Therein**

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Abstract Beyond its obvious macro-economic relevance, fiat money has important micro-economic implications. They matter for addressing No. 8 in Smale’s “Mathematical Problems for the Next Century” (Smale (1998)): extend the mathematical model of general equilibrium theory to include price adjustments. In the canonical Arrow-Debreu framework, equilibrium prices are set by a fictitious auctioneer. Removing that fiction raises the question of how prices are set and adjusted by decentralised actors with incomplete information. We investigate this question through a very basic model where a unique factor of production, labour, produces a single consumption good, called jelly for brevity. The point of the model is to study a price dynamics based on the firm’s expectations about jelly demand and labour supply. The system tends towards economic equilibrium, however, depending on the initial conditions it might not get there. In different model versions, different kinds of money are introduced. Compared to the case of no money, the introduction of money as a store of value facilitates the system reaching economic equilibrium. If money is introduced as a third commodity, i.e. there is also a demand for money, the system dynamics in general becomes more complex.

Keywords non-equilibrium price dynamics · expectations · agent-based modelling · macro-economic models

JEL Codes D52 · D84 · E12 · E40

1 Introduction

In the Arrow-Debreu framework (Arrow and Debreu (1954)), an auctioneer establishes equilibrium prices. In intertemporal models, these prices may display all sorts of dynamics, always brought about by the auctioneer. Unfortunately, the auctioneer is an auxiliary entity not to be found in actual economies. However, attempts to remove the auctioneer from economic models have consistently run in the unsolved question of how to understand and model non-equilibrium price dynamics.

In a non-equilibrium situation, usually it is assumed that prices somehow adjust according to excess demand ($\dot{p} \propto \xi(p)$, see e.g. Saari (1995)). But making this assumption, still the system dynamics can become arbitrarily complex, and it is not clear whether a stable equilibrium will be obtained, a fact which is known as the Sonnenschein-Mantel-Debreu (SMD) Theorem (Sonnenschein (1972), Mantel (1974), Debreu (1974)). An overview of the problems arising from the SMD Theorem is given in Rizvi (2006).

In the real world, when there is what is usually called a free market, the dynamics of prices result from the behaviour of a large number of different actors. Theoretically, such a system can be simulated using an agent-based model with many agents following their decision rules. An important attempt of understanding the dynamics of general equilibrium by agent-based modelling can be found in Gintis (2007). In such models, agents can have incomplete information and take decisions based on expectations (e.g. An et al (2007)). For modelling price dynamics it is not too easy a challenge to model the behaviour of the agents

in detail (since it is also not known in detail how real agents behave), and thus usually very rough assumptions are made, supposing that agents try to optimise their own utility. On the other hand, in many cases the general-equilibrium picture of clearing markets seems to be a good representation of what is happening in the real world. So the agent-based model allowing us to abandon the auctioneer should still be able to deliver these equilibrium results. The additional value of such an agent-based model is then supposed to be a better understanding of (real world) situations where an economic equilibrium is not obtained or disturbed (Farmer and Foley (2009)).

In this paper, the approach for getting rid of the auctioneer and modelling an out-of-equilibrium price-dynamics is the following: A price setting entity observes excess demand at the current price and updates the price for the next period accordingly. Let's assume that the price setting party seeks economic equilibrium but does not have complete information (because in that case it would be the auctioneer). Observation of some previous price and excess demand pairs $(p, \xi(p))$ does not suffice to determine the next price. Additional assumptions have to be made, and since there is no further insight into the future decisions of other actors, these additional assumptions of the price setter have to be grounded in expectations about the aggregate supply and demand. But this means that the present state of the system is influenced by expectations about the future. Linking the present state with expectations of the future, however, is what Keynes identified as one of the essential characteristics of money: "Or, perhaps, we might make our line of division between the theory of stationary equilibrium and the theory of shifting equilibrium – meaning by the latter the theory of a system in which changing views about the future are capable of influencing the present situation. For the importance of money essentially flows from its being a link between the present and the future." (Keynes (1936), Chapter 21) In this work, we investigate how the existence and amount of money in our very basic model influences the system's dynamics. Although here the availability of money is not taken into account when expectations are formed, we see that it limits the possibilities of the system to evolve over time. The amount of money thus influences which state the system converges to, i.e. whether starting out of equilibrium an economic equilibrium is obtained eventually.

For understanding some very fundamental principles, we intended to use a model as simple as possible:

- There are two aggregate agents and two goods. Good prices are set by one of the agents, with the other one reacting to the proposed price. The agents interact repeatedly. In our model, the agents are an aggregate firm that produces a consumption good (for brevity called jelly) from labour, and an aggregate household that supplies labour and consumes jelly.
- The agents have preferences/decision rules that determine their supply and demand of the different goods. The aggregate firm optimises its profit and the aggregate household its utility to determine supply and demand.
- The price setting agent has expectations about the relation of prices and aggregate supply/demand. In our model, prices are set by the firm.
- The price setting agent updates his expectations using observations about the actual excess demand at the prices that have been set before.

Figure 1 gives an overview of the agents and their interactions. There are three versions of the model which differ in whether and how money is part of the economic system.

The remainder of the paper is organised as follows: In Section 2 the agents are described in detail, and it is explained how transactions between agents take place. The three different model versions are discriminated. In Section 3 the dynamics of the system is discussed, and it is analysed which are the possible states (equilibria) it converges to. Section 4 shows some simulations illustrating the findings of Section 3. Sections 3 and 4 only deal with the first two versions of the model in which – although there might be something like "auxiliary money" – labour serves as numéraire. In Section 5 the third version of in the model is introduced. Here money has the role of a third commodity (and is the numéraire). Section 6 concludes the paper.

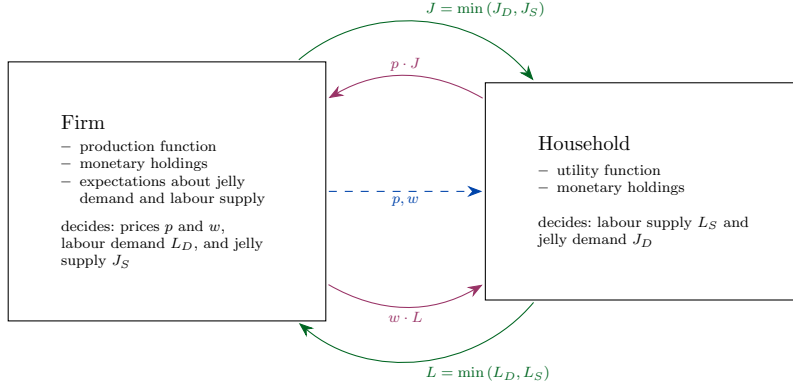


Fig. 1 Model overview

2 Agents, Markets, and Money

2.1 Household

The objective of the household is to maximise its utility subject to its budget constraint. Here, we have chosen a standard Cobb-Douglas utility function $U(L, J)$ depending on labour L and jelly consumption J :

$$U(L, J) = (L_f - L)^\alpha \cdot J^\beta \quad (1)$$

L_f is the maximum amount of labor force available. As usual, households are assumed to maximise $L_f - L$, interpreted as leisure time, and consumption J subject to a budget constraint. In the versions of the model we focus on in this paper¹, the budget constraint of the household is given by

$$p \cdot J = w \cdot L. \quad (2)$$

p is the price of jelly, w the wage. Equation (2) means that the household plans to spend its total income to buy jelly.

Maximising the utility given in (1) subject to the budget constraint (2) yields a utility maximum at

$$L_H = \frac{\beta}{\alpha + \beta} \cdot L_f \quad (3)$$

$$J_H = \frac{w}{p} \cdot \frac{\beta}{\alpha + \beta} \cdot L_f \quad (4)$$

The Cobb-Douglas form of (1) together with constraint (2) implies that L_H only depends on α, β, L_f and not on the wage-to-price ratio w/p . Indifference curves have the form

$$\sigma_c(L) = \frac{c}{(L_f - L)^{\frac{\alpha}{\beta}}} \quad (5)$$

In Figure 2, utility maxima are shown for different wage-to-price ratios $\frac{w}{p}$.

¹ These are the two of three versions, for a discrimination see Section 2.3. The household's utility optimisation for the third version is given in Section 5.1.1

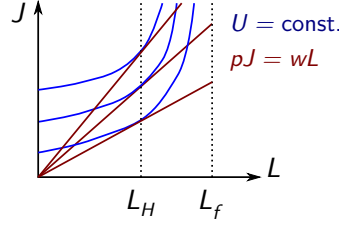


Fig. 2 Budget lines, $J = \frac{w}{p}L$, for different w - p -ratios (red) and indifference curves, $J = \sigma_c(L)$, for different constants c (blue) in the L - J -plane. The maxima of U subject to $p \cdot J = w \cdot L$ all lie on the vertical at $L = L_H$.

2.2 Firm

2.2.1 Profit Optimisation

In a situation of perfect competition of many firms, the aggregate firm must realise zero profits (von Neumann (1945)). Thus, maximisation of actual profits by individual firms results in minimisation of absolute profits by the aggregate firm. Therefore, the aggregate firm minimises the absolute value of the difference of revenues and expenditures:

$$\min |pJ - wL| \quad (6)$$

In most of what is discussed in this paper, labour serves as numéraire, i.e. $w = 1$. The production function is assumed to be $J = \rho(L) = L^\gamma$ (with $0 < \gamma < 1$). Let's assume that the firm's demand expectations can be represented by a function ϕ (that means at a jelly price of $\phi(J)$ the firm expects the jelly demand to be J). The optimisation problem becomes

$$\min_L |pJ - L| \quad (7)$$

$$\text{s.t. } J = \rho(L) = L^\gamma \quad (8)$$

$$p = \phi(J) \quad (9)$$

$$0 \leq L \leq L_f, \quad (10)$$

that means it can be written as

$$\min_{0 \leq L \leq L_f} |\rho(L) \cdot \phi(\rho(L)) - L|. \quad (11)$$

In the model, the firm optimizes its expected profit to determine prices, planned labour demand, and planned jelly supply. The firm's planned labour demand is the argument of the profit optimisation given in (11). Whether it is also the actual labour demand depends on whether the firm can afford to employ as much labour as it would like to, as explained in Section 3.1. An extension of the firm's optimisation for the case that money serves as numéraire and thus wages have to be set by the firm as well is given in Section 5.1.2.

2.2.2 Expectations and Learning

The firm's expectations have standard textbook form. The firm assumes a falling demand curve for jelly (and, for the case that labour is not the numéraire, a rising supply curve for labour). The updates of its expectation functions are based on observations of the household's jelly demand (and, in the third version of the model, labour supply). At each time step, information about the actual supply and demand at the current price is obtained by the firm and this information is used to update the expectation function(s). However,

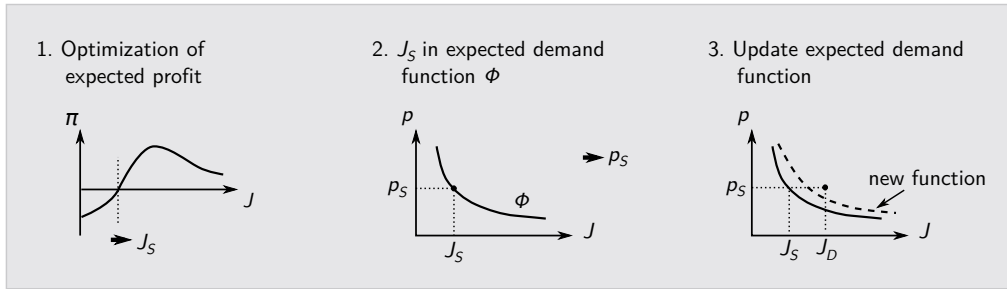


Fig. 3 Scheme of firm's learning algorithm: 1. Based on its demand expectations, the firm optimises its expected profit. This determines how much jelly J_S the firm wants to supply. 2. To set the price, the firm evaluates its expected demand function ϕ at $J = J_S$. The outcome $p_S = \phi(J_S)$ is set as jelly price. 3. At price p_S , the jelly demand J_D might be different from J_S . The firm uses this information to update its expected demand function. In the next time step, the firm uses this new expected demand function.

it is assumed that the firm also has a certain unwillingness to change its expectations essentially at every time step. A schematic overview of the algorithm used to update the expected labour demand function is given in Figure 3.

The updating algorithm adjusts the parameters for the expected demand (and supply) functions recursively. It approximates the last two observations. For the functional forms of ϕ and the updating algorithm used in the below simulations see the appendix.

2.3 Markets and Money

Having introduced the agents and their decision rules, the question arises whether and how the agents can fulfill their consumption/production plans. For producing/consuming they need to interact mutually which is – as usual – supposed to happen on markets. The fact that the model allows for non-equilibrium states requires rules for what happens for non-clearing markets.

In an exchange economy in equilibrium, transactions in all markets can be thought of as taking place instantaneously (although a temporal order may be natural, as e.g. in the case that labour has to be hired in order to produce a good sold in another market). All actors optimise their respective objective functions and prices are set by the auctioneer in a way that all markets clear. This means that all the agents' plans of how much of all goods to buy or sell can be fulfilled.

On the other hand, in a non-equilibrium situation, unmet supply or demand in one market can make it impossible for actors to follow their plans in other markets like in the above production example: if a producer is not able to hire as much labour as she has planned, she may not be able to produce as much as she had planned and thus her supply in the production good market may differ from her planned supply. This illustrates that for a non-equilibrium situation the temporal order of transactions in the different markets matters. However, the exchange of goods is usually thought of as an instantaneous act, otherwise the value of the good changing owners first has somehow to be stored by the seller until he buys the other good for this stored value. This means that value has to be stored at least for a short time. It suggests the introduction of (something like) money as a store of value for non-equilibrium situations.

In particular, in our model there are two markets, a labour and a jelly market, with the following features:

- In both markets, if supply does not equal demand, the short side of the market prevails.
- In every time step, first transactions take place in the labor market and later in the jelly market.

- In both markets, the buyer is limited by her budget, i.e. there is an upper limit of how much she can afford. If there is money, this upper limit is given by her cash balance. If there is no money, things become a bit trickier: for the household (buyer in the jelly market) the upper limit is given by the value of labour it sold at the last transaction in the labour market. For the firm (buyer in the labour market) the budget is given by the value of jelly it sold at the last transaction in the jelly market.

Introducing money in the system makes the system more complex. For example, it raises the questions of supply and demand for money, and if there is a demand for money, this means that more expectations are introduced to the model². However, the fact that we need to store value - at least for the short time between transactions in the two markets - suggests the introduction of some kind of money. In this paper we focus on two versions of the model, one without money and one with something like “light” money:

1. First version (no money): There is no money, only jelly and labour of equal value can be exchanged (quasi-)instantaneously, that means in the implementation two transactions are executed one after another (thus the value of labour or jelly can be stored for a short period of time, i.e. from the point of time transactions take place in one market to the point they take place in the other market).
2. Second version (money version 1): “Money” exists as a store of value, and in the beginning firm and household may have a certain amount of money at their disposal. However, for the household’s utility optimisation money is irrelevant, as well as in the firm’s profit optimisation. That means that there is no demand for money. Labour is still the numéraire, and the price of money equals one, i.e. one unit of money always has the same value as one unit of labour. Money is used to store value in units of labour.

At the end of the paper, with a third model version we give an outlook of how to modify the money “light” to make it become more similar to real money:

3. In the third version (money version 2) it is assumed that the household wishes to keep a fraction s of its current income in cash (e.g. to cover unforeseen costs in the future). This is implemented by changing the boundary condition in the household’s utility optimisation, as given in equation (16). This means that a demand for money is created. Money serves as numéraire, the price of labour may be different from one. The firm sets the wage based on its expectations about the labour supply in the same way as the jelly price is set based on expectations about the jelly demand (see Section 2.2.2). Thus, in money version 2, the two-commodity model is transformed into a three-commodity model.

In the following (Sections 3 and 4), we focus on the first two cases. A discussion of money version 2 is given in Section 5.

3 System Dynamics

Often it is assumed that economies are usually in economic equilibrium (all markets clear), and that this is a stable state. That means if they are pushed out of it they will come back to it quickly. This common assumption, however, is not part of general equilibrium theory (see e.g. Sonnenschein (1972)). So when we look at the dynamics of our simple model we are particularly interested in whether an initial out-of-equilibrium state of the system always converges to an economic equilibrium. But firstly, we still have to define rules for how our systems evolves over time which is done in the first part of this section. In the second part it is discussed how the economic equilibria look like in our model and the last part addresses the question just raised, that is whether the systems obtains economic equilibrium eventually.

² The demand for money is based on the expectation that it will be useful at some later point in time.

3.1 Evolution Rules

If an aggregate household and firm interact along the lines indicated above, it is useful to consider an iterated cycle involving the following steps:

1. *Firm plans production*: The firm has expectations about the household's jelly demand and labour supply. By optimizing its expected profit the firm plans the jelly production and thus the labour demand, and sets the jelly price p accordingly. However, the labour demand L_D is limited by the firm's budget. In the version without money, the budget limitation is given by the revenues of the last jelly transaction, in the case that there is money by the firm's monetary holdings.
2. *Household plans consumption*: Optimising its utility function subject to its budget constraint the household decides how much labour L_S to offer and how much jelly it plans to consume given price p .
3. *Labour market transactions*: In the labour market, the short side of the market prevails, i.e. $L_M = \min\{L_D, L_S\}$.
4. *Firm produces jelly*: Using L_M the firm produces the actual jelly supply J_S . If $L_M = L_D$, J_S equals the planned supply but if $L_M < L_D$, J_S is less than the planned production.
5. *Household decides jelly demand*: If $L_M = L_S$ the household's jelly demand J_D equals the planned consumption, if $L_M < L_S$ the household demands as much jelly as it can afford. If there is no money, this always means $J_D = w/p \cdot L_M$, with money the jelly demand might be different since the household can also use money from the last period to acquire jelly.
6. *Jelly market transactions*: In the jelly market, the short side of the market prevails, i.e. $J_M = \min\{J_D, J_S\}$.
7. *Expectations update*: Using the transaction information (actual jelly demand J_D at price p) the firm updates its jelly demand expectation function.

With money, the cycle is close to the intuition of everyday life in a modern economy. Where supply and demand do not match, the difference will affect the money holdings of the agents. If the firm can hire less labour than it intended, it will remain with more money than it planned, if the household can sell less labour than it intended, it will remain with less money than it planned. The analogous pattern arises with sales of jelly.³

With money the monetary holdings are a constraint for what an agent can plan to buy and also for what she may actually buy when plans cannot be realised. Without money we set an analogous constraint. The labour sold by the household in step 3 then defines a budget constraint for step 5, and the jelly sold by the firm in step 5 defines a budget constraint for step 3 in the subsequent cycle.

It might be useful to point out that in this paper we use the term *budget constraint* for the budget constraint the household takes into account when optimising its utility, given in equation (2). On the other hand, we use the term *budget limitations* for the limitations household and firm might experience when they want to meet their demands but their monetary holdings, or for the case without money their revenues from the last market interaction, do not suffice to do that completely (see steps 1 and 5). Budget limitations are not taken into account in the respective optimisations.

3.2 Economic Equilibria

The term equilibrium is used in different ways, for our purposes it is useful to clearly differentiate between economic equilibria, i.e. states of the system where supply equals demand in all markets, and dynamic equilibria, i.e. fixed points of the dynamic system. Of course,

³ Of course, monetary holdings can also change when markets clear. The difference is that if they do not clear, the monetary holdings must be affected.

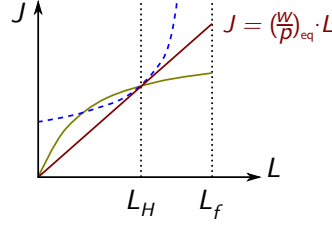


Fig. 4 Equilibrium wage-price-ratio $(\frac{w}{p})_{\text{eq}}$.

economic equilibria can be fixed points, and vice versa, but this is not necessarily the case.

The system being studied here has only one economic equilibrium. Labour serves as numéraire, so $w = 1$. The household's utility maximisation subject to its budget constraint yields a maximum at

$$L_H = \frac{\beta}{\alpha + \beta} \cdot L_f \quad (12)$$

$$J_H = \frac{1}{p} \cdot \frac{\beta}{\alpha + \beta} \cdot L_f \quad (13)$$

(see Section 2.1). At the economic equilibrium the labour supply $L_S = L_H$ and the jelly demand $J_D = J_H$. In the L - J -plane (L_H, J_H) is the point where the budget line $J = w/p \cdot L$ intersects with the vertical at $L = L_H$. For the firm, in the L - J -plane the planned production is represented by the point of intersection of the budget line $J = w/p \cdot L$ and the production function $J = L^\gamma$. Economic equilibrium requires supply to equal demand in both markets, i.e. for equilibrium the budget line has to intersect with the production function at $L = L_H$ (see Figure 4). Setting again $w = 1$, thus $J_H = (L_H)^\gamma$ determines the equilibrium price p_E :

$$\frac{1}{p_E} = \left(\frac{\beta}{\alpha + \beta} L_f \right)^{\gamma-1}. \quad (14)$$

3.3 Dynamic Equilibria

Dynamic equilibria are the fixed points (steady states) of the system, i.e. the points in state space that once the system has reached them it does not leave them any more. However, what we are interested here is rather not the evolution of all state variables (e.g. the parameters of the firm's expected demand function which don't really have a "real world" equivalent) but only the evolution of the variables that matter economically, which are supplies, demands, and prices. We suppose that our economy has reached a fixed point if these "economic" variables do not change any more over time.

Definition 1 (Economic space, economic states) Labour supply L_t^S and demand L_t^D , jelly supply J_t^S and demand J_t^D , and the jelly price p_t at time t are state variables of the system. The space spanned by these variables is called the economic space $\mathcal{E} \subseteq \mathbb{R}_+^5$. Let $s_t = (L_t^D, L_t^S, J_t^D, J_t^S, p_t, \dots)$ be the state of the system at time t , i.e. the vector of all state variables. Then the projection of the state s_t on the economic space, i.e. the vector $e_t = (L_t^D, L_t^S, J_t^D, J_t^S, p_t) \in \mathbb{R}_+^5$ is called the economic state of the system at time t .

The system under study is deterministic. This means that state space trajectories do not intersect. To know that for time $t = 0$ the system is at a certain point s_0 in state space suffices for knowing the systems behaviour for all $t > 0$. Since some of the necessary

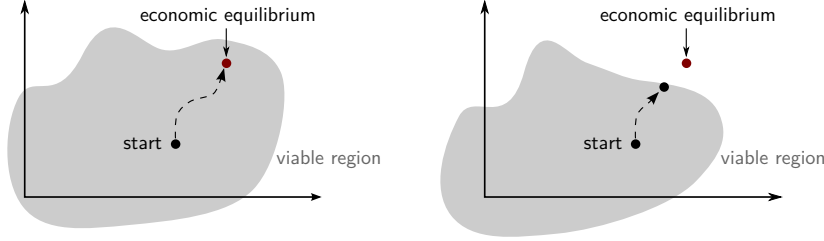


Fig. 5 Economic space with budget limitations: If the *learning trajectory* (broken line) towards the economic equilibrium lies completely inside the viable region, the economic equilibrium is obtained eventually (left). If not, a dynamic equilibrium (a *border equilibrium*) is reached at the border of the viable region (right).

information is lost projecting states onto economic states, this is not true for trajectories in the economic space⁴. As explained above, we are interested in the dynamic equilibria in the economic space, and in whether they are economic equilibria as well. It turns out, that while the economic equilibrium is also dynamic equilibrium, also other dynamic equilibria can be found in which not all the markets clear.

Proposition 1 (Types of fixed points) *In the economic space of the system, there are two types of fixed points, the economic equilibrium and one other type which will be called “border equilibrium”. In a border equilibrium, at least one market does not clear.*

To understand this, in the following we consider how trajectories in the economic space look like and how the agents’ budget limitations restrict them.

Definition 2 (Learning trajectory) The state space learning trajectory $\{s_t\}$ of an initial state $s_{\text{in}} = s_0$ is the sequence of states $\{s_t\}$ that are obtained after t time steps if the system’s evolution rules are applied without any budget limitations. The learning trajectory $\{l_t\}$ is the projection of $\{s_t\}$ onto the economic space.

Proposition 2 *All learning trajectories converge to the economic equilibrium e_E , i.e. $\lim_{t \rightarrow \infty} l_t = e_E$ for all initial states s_{in} .*

Proposition 2 means that the firm’s learning mechanism directs the system towards the economic equilibrium. However, the learning trajectory does not take into account budget limitations. At each time step, some regions of the state space are not available, e.g. because the firm’s budget does not suffice to buy the respective labour.

Definition 3 (Viable region) At time t , the set of all economic states that could be reached at time $t + 1$ without violating the budget limitations of an agent is called the viable region at time t . An element of the viable region is called viable at time t .⁵

The statement of Proposition 1 is sketched in Figure 5. There can be two possible situations: 1. The learning trajectory lies completely in the viable region at all times t . Then, as Proposition 2 states, the economic equilibrium is obtained eventually. 2. The learning trajectory does not lie completely in the viable region. Then the system evolves until it reaches the boarder of the viable region and stays there.

The viable region can vary over time depending on how the budget limitations of the agents develop over time.

⁴ The system’s evolution in the economic space can be considered as the evolution of a non-deterministic system with the source of the uncertainty resulting from the lack of knowledge of some “hidden variables” (like e.g. the parameters z_t, ζ_t of the firm’s expected demand function $\phi_t(J) = \frac{z_t}{J\zeta_t}$).

⁵ An association with viability theory (e.g. Aubin (1991)) is intended although the detailed discussion of how it is connected with this work is not part of the paper.

Proposition 3 *If there is no money, economic states that are not viable in the beginning cannot become viable later, i.e. the viable region can only shrink over time.*

Proof: If there is no money, the firm's budget $B_{t+1}^{(F)}$ is given by the amount of jelly J_t sold at time t times the jelly price p_t , i.e. $B_{t+1}^{(F)} = p_t J_t$. The household's budget is its current income, i.e. $B_t^{(H)} = L_t$ (since $w_t = 1$). That means the household's budget limitation is $p_t J_t \leq L_t$, and the firm's budget limitation is $L_{t+1} \leq p_t J_t$. From these two inequalities follows $p_{t+1} J_{t+1} \leq p_t J_t$ and $L_{t+1} \leq L_t$, i.e. $B_{t+1}^{(F)} \leq B_t^{(F)}$ and $B_{t+1}^{(H)} \leq B_t^{(H)}$. But if the budgets of both agents cannot become larger, neither does the viable region. \square

Definition 4 (Viability closure) For a system with a constant total amount of money $M = m_t^{(H)} + m_t^{(F)}$, the set of economic states that are viable when both agents have M at their disposal at the time they make decisions about supply, demand, and prices, is called viability closure.

Proposition 4 *In a system with money, the viability closure is constant over time and the viable region is a subset of the viability closure.*

Proof: The viability closure is constant over time since the total amount of money in the system $M = m^{(H)} + m^{(F)}$ is constant over time. Since the firm's budget $B_t^{(F)} = m_t^{(F)} \leq M$ and the household's budget $B_t^{(H)} = m_t^{(H)} \leq M$ all viable states at time t lie in the viability closure. \square

With "full" money, on the other hand, it is not so clear how the viable region changes over time as money can be redistributed among agents. However, there is an upper limit for the size of the viable region, given by the (constant) total amount of money in the system.

Proposition 5 *For a system with total amount of money M and economic-equilibrium labour L_E , if $M < L_E$ in the beginning, the economic equilibrium can not be reached.*

Proof: To employ L_E units of labour at time t , the firm needs to be able to pay them, thus $m_t^{(F)} \geq L_E$ has to hold. But if $M < L_E$ this is not possible because $m_t^{(F)} \leq M$ and thus $m_t^{(F)} < L_E$. \square

4 Simulations

We programmed a model with the characteristics described above to illustrate the dynamic evolution of such a model discussed in the previous section. As simulation outcome, we are particularly interested in whether the system finally obtains the economic equilibrium or not, which can be observed by either comparing supply and demand on both markets or by recording the final utility.

4.1 Final Utility Versus Initial Expectations With and Without Money

Figure 6 shows the utility at the fixed point the system converges to (utility after 50 time steps) versus the initial parameter of the firm's expected demand function. In Figure 6a) simulations without money are shown, where the initial revenue (at $t = 0$) of the firm is supposed to be exactly what it would be at the economic equilibrium (i.e. the economic equilibrium is viable at $t = 1$). The firm's initial expectations are characterised by ζ_0 and Δ_z . The firm's expected demand function is given by

$$\phi_t(J) = \frac{z_t}{J\zeta_t}. \quad (15)$$

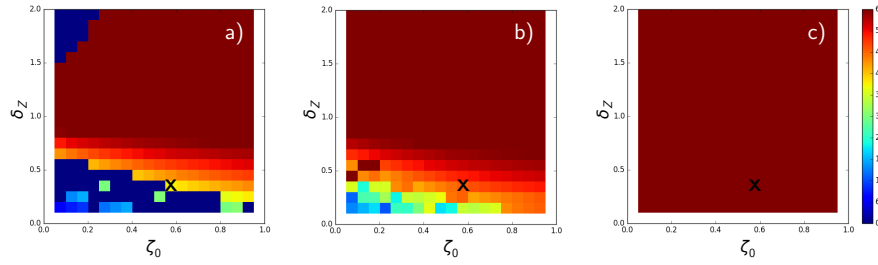


Fig. 6 Utility after 50 time steps versus initial demand expectations of the firm, ζ_0 and Δ_z . a) No money and “right” initial amount of revenue by the firm. b) With money; initial amount of firm corresponds to initial revenues in a), household with no initial money. c) Firm and household have a large amount of money initially. Temporal evolution of the economic variables at $\zeta_0 = 0.55$ and $\Delta_z = 0.3$ (marked point) are shown Figures 7, 8, and 9.

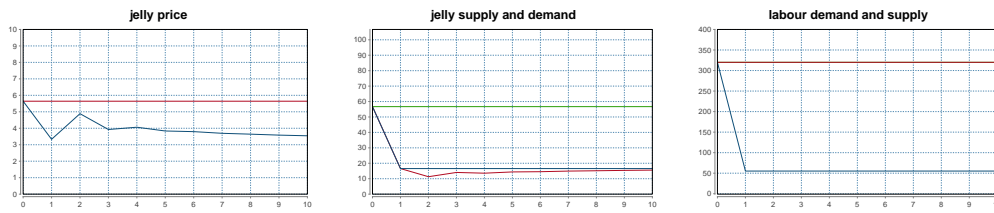


Fig. 7 Temporal evolution of system with initial expectations $\zeta_0 = 0.55$ and $\Delta_z = 0.3$ from Figure 6a (border equilibrium). From left to right: a) Jelly price (blue) and price at economic equilibrium (red). b) Jelly supply (blue), demand (red), and demand and supply at economic equilibrium (green). c) Labour demand (blue) and supply (red); the labour supply also equals demand and supply at economic equilibrium.

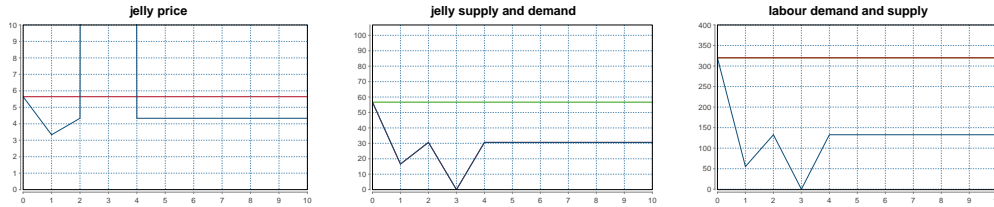


Fig. 8 Temporal evolution of system with initial expectations $\zeta_0 = 0.55$ and $\Delta_z = 0.3$ from Figure 6b (border equilibrium). From left to right: a) Jelly price (blue) and price at economic equilibrium (red). b) Jelly supply (blue), demand (red, under blue line), and demand and supply at economic equilibrium (green). c) Labour supply also equals demand and supply at economic equilibrium.

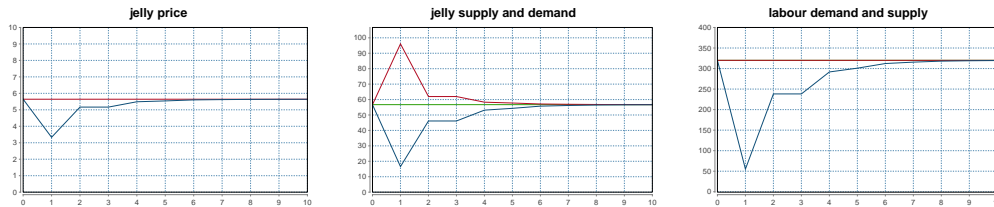


Fig. 9 Temporal evolution of system with initial expectations $\zeta_0 = 0.55$ and $\Delta_z = 0.3$ from Figure 6c (economic equilibrium). From left to right: a) Jelly price (blue) and price at economic equilibrium (red). b) Jelly supply (blue), demand (red, under blue line), and demand and supply at economic equilibrium (green). c) Labour demand (blue) and supply (red); the labour supply also equals demand and supply at economic equilibrium.

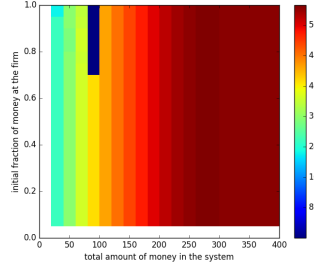


Fig. 10 Utility after 50 time steps versus amount of total money in the system M and initial money distribution. The initial money distribution hardly matters, but the total amount of money is important. For $M \geq L_E = 320$, the economic equilibrium is obtained.

ζ_0 is the initial value for ζ . Let p_E and J_E be jelly price and jelly demand at the economic equilibrium. δ_z characterises how much the initial z_0 deviates from the value $\tilde{z} = p_E \cdot J_E^{\zeta_0}$, i.e. from the value for which, given ζ_0 , $\phi_0(J_E) = p_E$. So the initial z_0 is given by $z_0 = p_E J_E^{\zeta_0} \cdot \delta_z$.

Figure 6b) also shows the utility after 50 time steps versus initial expectations ζ_0 and Δ_z , but this time with money: the initial cash balance of the firm equals the firm's revenues at economic equilibrium, i.e. in the first time step the firm can buy as much labour as in Figure 6a), and the economic equilibrium is as well viable at time $t = 1$. The initial cash balance of the household is zero, so the initial situation is quite similar to the one of Figure 6a), only that there is money which can store value over time. The final utility looks similar to Figure 6a), however, for some ζ_0 and Δ_z combinations the final utility is higher and the system reaches the economic equilibrium more often.

For the simulations of Figure 6c) firm and household have a large amount of money at their disposal initially: $m_0^{(F)} = m_0^{(H)} = 400$ (which is equivalent to the labour force, since $L_f = 400$ in all the simulations). So there are basically no budget limitations, and as stated by Proposition 2 the system converges to the economic equilibrium for all initial expectations ζ_0 , Δ_z .

Figures 7, 8, and 9 show the temporal evolution of prices, and jelly and labour market for $\zeta_0 = 0.55$ and $\Delta_z = 0.3$, i.e. the point marked in Figures 6a-c.

4.2 Initial Total Money and Initial Money Distribution

For the case with money, Figure 10 shows the final utility versus the total amount of money M in the system and the initial money distribution. In the simulation $L_E = 320$. It can be seen that, as stated by Proposition 5, if $M < L_E$ the economic equilibrium is not obtained. In contrast to the total amount of money M in the system, the initial money distribution seems to be quite irrelevant for the long term equilibrium. But in some cases it can make a difference: Figure 11 shows the same simulation as Figure 8 with the only difference that the firm does not possess all the money initially but around 5% of it is with the household. Unlike in the simulation from Figure 8, here the economic equilibrium is obtained.

5 Model Extension: Money As a Third Commodity

The three versions of our model (see Section 2.3) differ in whether and how money is part of the economic system. Up to now, only the dynamics of the first two of them has been discussed (Sections 3 and 4). While in the first version there is no money at all, in the second version there is – but neither household nor firm have a demand for it. It does not play a role when agents do their optimisations to plan supply and demand of labour and

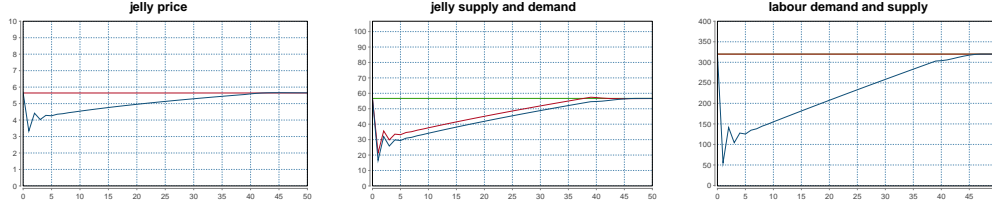


Fig. 11 Temporal evaluation as in Figure 8 with the only difference that the firm does not possess all the money initially but about 5% is initially with the household. From left to right: a) Jelly price (blue) and price at economic equilibrium (red). b) Jelly supply (blue), demand (red), and demand and supply at economic equilibrium (green). c) Labour demand (blue) and supply (red); the labour supply also equals demand and supply at economic equilibrium.

jelly. Only when it comes to whether their plans can be fulfilled their cash balances may be important.

In this Section, the third version is discussed: it is assumed that the household wants to keep a certain amount of its income in cash. That means that a demand for money is created. But as soon as there is a demand for money, money has a price and can serve as numéraire.

The evolution rules for the extended system are the same as before (see Section 3.1). Only here, when the firm sets the jelly price it also sets the wage, and when updating its expected jelly demand function it also updates its expected labour supply.

5.1 Agents' Optimisations

For determining supply and demand, the household optimises its utility and the firm its profit. Here, there are two changes compared to the previous versions (Sections 2.1 and 2.2.1), firstly, the household's budget constraint takes into account the preference of having a certain amount of money in cash (as described in Section 5.1.1) and secondly, since money now serves as numéraire, the firm has to set a wage which is done based on its labour supply expectations (Section 5.1.2).

5.1.1 Household

The household is assumed (to plan) to keep a fraction s of its current income in cash (e.g. to cover unforeseen costs). The budget constraint becomes

$$\begin{aligned} pJ + s \cdot wL &= wL + m^{(H)}, \quad \text{that is} \\ pJ &= (1 - s) \cdot wL + m^{(H)}. \end{aligned} \quad (16)$$

with $m^{(H)}$ being the monetary holdings of the household from the previous time period. Maximising the utility given in (1) subject to this budget constraint (16) yields a utility maximum at

$$L_H = \frac{\beta}{\alpha + \beta} \cdot L_f - \frac{\alpha}{\alpha + \beta} \frac{m^{(H)}}{(1 - s) \cdot w} \quad (17)$$

$$J_H = \frac{\beta}{\alpha + \beta} \cdot \left((1 - s) \frac{w}{p} L_f + \frac{m^{(H)}}{p} \right) \quad (18)$$

5.1.2 Firm

The firm's profit optimisation (see Section 2.2.1) has to be adapted as well: With money being the numéraire and assuming ψ to represent the firm's expectations about the labour supply (at wage $\psi(L)$ the firm expects the labour supply to be L) it is given by

$$\min_L |pJ - wL| \quad (19)$$

$$\text{s.t. } J = \rho(L) = L^\gamma \quad (20)$$

$$p = \phi(J) \quad (21)$$

$$w = \psi(L) \quad (22)$$

$$0 \leq L \leq L_f, \quad (23)$$

that is

$$\min_{0 \leq L \leq L_f} |\rho(L) \cdot \phi(\rho(L)) - L \cdot \psi(L)|. \quad (24)$$

5.2 Economic Equilibria

If money is seen as a third commodity, economic equilibrium would mean that not only in the labour and jelly market, but also in the money market supply equals demand. However, in this section we only examine economic equilibria in the same sense as for the previous model versions, i.e. we suppose the economy to be in equilibrium if $L_t^{(S)} = L_t^{(D)}$ and $J_t^{(S)} = J_t^{(D)}$, and then we are interested in whether these points are fixed points in the economic space (as defined above) and whether there are other fixed points.

Economic equilibria can then be derived as follows. The household's utility maximisation subject to its budget constraint yields a maximum at

$$L_H = \frac{\beta}{\alpha + \beta} \cdot L_f - \frac{\alpha}{\alpha + \beta} \frac{m^{(H)}}{(1-s) \cdot w} \quad (25)$$

$$J_H = \frac{\beta}{\alpha + \beta} \cdot \left((1-s) \frac{w}{p} L_f + \frac{m^{(H)}}{p} \right) \quad (26)$$

(see Section 5.1.1). Necessary condition for an economic equilibrium is $J_H = L_H^\gamma$, i.e.

$$\frac{\beta}{\alpha + \beta} \cdot \left((1-s) \frac{w}{p} L_f + \frac{M}{p} \right) = \left(\frac{\beta}{\alpha + \beta} \cdot L_f - \frac{\alpha}{\alpha + \beta} \frac{m^{(H)}}{(1-s) \cdot w} \right)^\gamma. \quad (27)$$

For a dynamically stable equilibrium, however, also $m^{(H)} = swL$ has to hold (otherwise the household would change its labour supply and jelly demand for the next time step), that means for the equilibrium labour supply L_E

$$L_E = \frac{\beta}{\alpha + \beta} \cdot L_f - \frac{\alpha}{\alpha + \beta} \cdot \frac{s}{(1-s)} \cdot L_E, \quad \text{and thus} \\ L_E = \frac{\beta(1-s)}{\alpha + \beta(1-s)} L_f. \quad (28)$$

For the respective planned jelly demand J_E then

$$J_E = \frac{w}{p} \cdot \frac{\beta(1-s)}{\alpha + \beta(1-s)} L_f = \frac{w}{p} L_E. \quad (29)$$

Setting $J_E = L_E^\gamma$ yields for the (temporarily stable) economic equilibrium wage-to-price ratio

$$\left(\frac{w}{p} \right)_E = \left(\frac{\beta(1-s)}{\alpha + \beta(1-s)} L_f \right)^{\gamma-1}. \quad (30)$$

Depending on w, p, M there might be other economic equilibria if $J_H = L_H^\gamma$ but (30) holds for the ones that are stable over time.

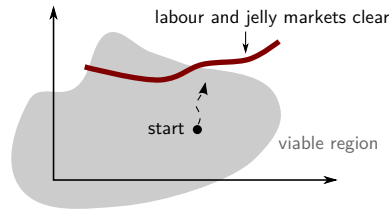


Fig. 12 With money as a third good, there is more than one point in the economic space where jelly and labour markets clear. These points can lie inside, outside or partly in the viable region.

5.3 Dynamic Equilibria, Comparison with Previous Cases

In the extended model, there is one more market and one more price (with the respective expectation-and-updating routine by the firm) and thus the resulting dynamics of the system become more complex. A detailed analysis is beyond the scope of this paper and will be subject of further research. However, some similarities and differences to the previous cases are pointed out in the following.

A couple of propositions have been made about the dynamics of the original model. For the extended model, first simulations suggest that the same types of fixed points can be found for the extended model, so it seems that Proposition 1 applies for the extended model as well.

However, there is a fundamental difference between the extended model and the previous cases: equations (12)-(14) determine a single point in the economic space but equations (28)-(30) an infinite set of points, i.e. for the previous cases the economic equilibrium is a single point in the economic space but for the extended model it is a larger set. The set of economic equilibria can also lie partly in the viable region, as depicted in Figure 12.

This can be illustrated e.g. reformulating Proposition 5 for the extended model. This proposition states that (for the previous case with money) it is a necessary condition for the economic equilibrium that there is a minimum total amount of money $M = L_E$ in the system. For the extended model, the respective minimum amount of money would be $M = (1 + s)w \cdot L_E$ (there has to be enough money for paying the equilibrium income plus the amount the household keeps under the mattress), i.e. it depends on the wage level, whether the necessary condition for the total amount of money is fulfilled or not.

Of course, here we defined the economic equilibrium only by clearing labour and jelly markets while not paying attention to the money market. The full analysis of the three goods case (jelly, labour, money) is beyond the scope of this paper, and will be pursued in future work. But the findings of this paper already suggest that this analysis will focus on the question what kind of monetary policies make economic equilibria viable.

6 Conclusions

It is an open question in economic theory how to extend the mathematical model of general equilibrium theory to include price adjustments. Removing the fictitious auctioneer from the canonical Arrow-Debreu framework raises the question of how prices are set and adjusted by decentralised actors with incomplete information.

In this paper, we considered a very basic model with only two aggregate agents, a household and a firm. The firm sets the prices. It compensates for the lack of information about the household's demand using expectations about that. It improves its expectations about the household's demand by observing the household's reactions to past prices.

We investigated the dynamics of this system with respect to the question whether the economic equilibrium is obtained eventually. We found that it is essential for such a system that value can be stored over time. For this purpose, it seems to be natural to introduce

money to the modelled economy, but this raises questions about whether and to which degree money is a commodity of its own, and how to determine supply and demand for it.

One main point of this paper is that we look at the trajectories of the system in what we call the economic space, i.e. a projection of the state space onto the “economic” state variables: supplies, demands, and prices. While the system is deterministic, evolution in the economic space is not (i.e. only knowing the “economic” variables does not suffice to predict the future development). But statements can be made about which parts of the economic space are allowed for evolution, and the system’s “freedom” to evolve over time is strongly connected to the availability of money to the agents. The task to model an out-of-equilibrium price dynamics based on expectations brought up the need for money in the system “naturally”. Therefore, we consider this paper as a first step towards modelling an out-of-equilibrium economy with money and (at least) two other commodities.

Acknowledgements Various discussions with Andreas Geiges and Franziska Schütze have been very useful and are highly acknowledged, as well as valuable programming support by Steffen Fürst.

Appendix: The Dynamic System

In the following, the dynamic system is formulated for all three versions of the model.

1. Firm decides production

Versions 1 and 2, labour numéraire:

$$L_{t+1}^{(D)} = \arg \min_L |\rho(L) \cdot \phi_t(\rho(L)) - L| \quad (31)$$

$$\text{with } \rho(L) = L^\gamma \quad (32)$$

$$\text{and } \phi_t(\rho(L)) = \frac{z_t}{\rho(L)\zeta_t} \quad (33)$$

$$\text{and } 0 \leq L \leq L_f \quad (34)$$

$$\text{and } L \leq \begin{cases} p_t \cdot J_t^{(M)} & (\text{version 1}) \\ m_t^{(F)} & (\text{version 2}) \end{cases} \quad (35)$$

$$J_{t+1}^{(S, \text{planned})} = \rho(L_{t+1}^{(D)}) = (L_{t+1}^{(D)})^\gamma \quad (36)$$

$$p_{t+1} = \phi_t(J_{t+1}^{(S, \text{planned})}) \quad (37)$$

Version 3, money numéraire:

$$L_{t+1}^{(D)} = \arg \min_L |\rho(L) \cdot \phi_t(\rho(L)) - \psi_t(L) \cdot L| \quad (38)$$

$$\text{with } \rho(L) = L^\gamma \quad (39)$$

$$\text{and } \phi_t(\rho(L)) = \frac{z_t}{\rho(L)\zeta_t} \quad (40)$$

$$\text{and } \psi_t(L) = \left(\frac{x_t}{L_f - L} \right)^{\frac{1}{\xi_t}} \quad (41)$$

$$\text{and } 0 \leq L \leq L_f \quad (42)$$

$$\text{and } L \leq \arg \min_{L \geq 0} \left| \psi_t(L) - \frac{m_t^{(F)}}{L} \right| \quad (43)$$

$$J_{t+1}^{(S, \text{planned})} = \rho(L_{t+1}^{(D)}) = (L_{t+1}^{(D)})^\gamma \quad (44)$$

$$p_{t+1} = \phi_t(J_{t+1}^{(S, \text{planned})}) \quad (45)$$

$$w_{t+1} = \psi_t(L_{t+1}^{(D)}) = \left(\frac{x_t}{L_f - L_{t+1}^{(D)}} \right)^{\frac{1}{\xi_t}} \quad (46)$$

2. Household decides labour

Versions 1 and 2, labour numéraire:

$$L_{t+1}^{(S)} = L_f \cdot \frac{\beta}{\alpha + \beta} \quad (47)$$

Version 3, money numéraire:

$$L_{t+1}^{(S)} = L_f \cdot \frac{\beta}{\alpha + \beta} - \frac{\alpha}{\alpha + \beta} \cdot \frac{m_t^{(H)}}{(1-s) \cdot w_{t+1}} \quad (48)$$

3. Labour market transaction (all versions)

$$L_{t+1}^{(M)} = \min \left\{ L_{t+1}^{(D)}, L_{t+1}^{(S)} \right\} \quad (49)$$

4. Firm produces jelly (all versions)

$$J_{t+1}^{(S)} = (L_{t+1}^{(M)})^\gamma \quad (50)$$

5. Household decides consumption

Version 1, labour numéraire:

$$J_{t+1}^{(D)} = \frac{L_{t+1}^{(M)}}{p_{t+1}} \quad (51)$$

Versions 2, labour numéraire:

$$J_{t+1}^{(D)} = \min \left\{ \frac{\beta}{\alpha + \beta} \cdot \frac{L_f}{p_{t+1}}, \frac{m_t^{(H)}}{p_{t+1}} \right\} \quad (52)$$

Version 3, money numéraire:

$$J_{t+1}^{(D)} = \min \left\{ \frac{\beta}{\alpha + \beta} \cdot \left((1-s) \frac{w_{t+1}}{p_{t+1}} L_f + \frac{m_t^{(H)}}{p_{t+1}} \right), \frac{m_t^{(H)}}{p_{t+1}} \right\} \quad (53)$$

6. Jelly market transaction (all versions)

$$J_{t+1}^{(M)} = \min \left\{ J_{t+1}^{(D)}, J_{t+1}^{(S)} \right\} \quad (54)$$

7. Firm updates monetary holdings (only versions 2 and 3)

$$m_{t+1}^{(F)} = m_t^{(F)} + p_{t+1} \cdot J_{t+1}^{(M)} - w_{t+1} \cdot L_{t+1}^{(M)} \quad (55)$$

(with $w_{t+1} = 1$ in version 2)**8. Firm updates expected jelly-demand function** (all versions)

$$(\zeta_{t+1}, z_{t+1}) = \arg \min_{\zeta, z} \left(\sqrt{\frac{\ln^2 \Delta_{t+1}^{(p)} + \epsilon_1 \cdot \ln^2 \Delta_t^{(p)}}{1 + \epsilon_1}} + \epsilon_2 \sqrt{\ln^2 \frac{\zeta}{\zeta_t} + \ln^2 \frac{z}{z_t}} \right)$$

$$\text{with } \Delta_\tau^{(p)} = \frac{p_\tau}{\phi(J_\tau^{(D)})}$$

$$\text{and } z > 0, 1 > \zeta > 0$$

9. **Firm updates expected labour-supply function** (only version 3)

$$(\xi_{t+1}, x_{t+1}) = \arg \min_{\xi, x} \left(\sqrt{\frac{\ln^2 \Delta_{t+1}^{(w)} + \epsilon_1 \cdot \ln^2 \Delta_t^{(w)}}{1 + \epsilon_1}} + \epsilon_2 \sqrt{\ln^2 \frac{\xi}{\xi_t} + \ln^2 \frac{x}{x_t}} \right) \quad (56)$$

$$\text{with } \Delta_\tau^{(w)} = \frac{w_\tau}{\psi(L_\tau^{(S)})} \quad (57)$$

$$\text{and } x > 0, \xi > 0 \quad (58)$$

10. **Household updates monetary holdings** (only versions 2 and 3)

$$m_{t+1}^{(H)} = m_t^{(H)} - p_{t+1} \cdot J_{t+1}^{(M)} + w_{t+1} \cdot L_{t+1}^{(M)} \quad (59)$$

(with $w_{t+1} = 1$ in version 2)

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